

Very well-covered graphs and the unimodality conjecture

Vadim E. Levit and Eugen Mandrescu
 Department of Computer Science
 Holon Academic Institute of Technology
 52 Golomb Str., P.O. Box 305
 Holon 58102, ISRAEL

February 1, 2008

Abstract

If s_k denotes the number of stable sets of cardinality k in the graph G , then $I(G; x) = \sum_{k=0}^{\alpha(G)} s_k x^k$ is the *independence polynomial* of G (Gutman, Harary, 1983), where $\alpha(G)$ is the size of a maximum stable set in G . Alavi, Malde, Schwenk and Erdős (1987) conjectured that $I(T, x)$ is unimodal for any tree T , while, in general, they proved that for any permutation π of $\{1, 2, \dots, \alpha\}$ there is a graph G with $\alpha(G) = \alpha$ such that $s_{\pi(1)} < s_{\pi(2)} < \dots < s_{\pi(\alpha)}$. Brown, Dilcher and Nowakowski (2000) conjectured that $I(G; x)$ is unimodal for any well-covered graph. Michael and Traves (2002) provided examples of well-covered graphs with non-unimodal independence polynomials. They proposed the so-called "roller-coaster" conjecture: for a well-covered graph, the subsequence $(s_{\lceil \alpha/2 \rceil}, s_{\lceil \alpha/2 \rceil + 1}, \dots, s_{\alpha})$ is unconstrained in the sense of Alavi *et al.* The conjecture of Brown *et al.* is still open for very well-covered graphs, and it is worth mentioning that, apart from K_1 and the chordless cycle C_7 , connected well-covered graphs of girth ≥ 6 are very well-covered (Finbow, Hartnell and Nowakowski, 1993).

In this paper we prove that $s_{\lceil (2\alpha-1)/3 \rceil} \geq \dots \geq s_{\alpha-1} \geq s_{\alpha}$ are valid for any (a) bipartite graph G with $\alpha(G) = \alpha$; (b) quasi-regularizable graph G on $2\alpha(G) = 2\alpha$ vertices. In particular, we infer that this is true for (a) trees, thus doing a step in an attempt to prove Alavi *et al.*' conjecture; (b) very well-covered graphs. Consequently, for this case, the unconstrained subsequence appearing in the roller-coaster conjecture can be shorten to $(s_{\lceil \alpha/2 \rceil}, s_{\lceil \alpha/2 \rceil + 1}, \dots, s_{\lceil (2\alpha-1)/3 \rceil})$. We also show that the independence polynomial of a very well-covered graph G is unimodal for $\alpha(G) \leq 9$, and is log-concave whenever $\alpha(G) \leq 5$.

key words: *stable set, independence polynomial, unimodal sequence, quasi-regularizable graph, bipartite graph, very well-covered graph.*

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X . By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$. We also denote by $G - F$ the partial subgraph of G obtained by deleting the edges of F , for $F \subset E(G)$, and we write shortly $G - e$, whenever $F = \{e\}$.

A vertex v is *pendant* if its neighborhood $N(v) = \{u : u \in V, uv \in E\}$ contains only one vertex; an edge $e = uv$ is *pendant* if one of its endpoints is a pendant vertex. \overline{G} stands for the complement of G , while K_n, P_n, C_n denote respectively, the complete graph on $n \geq 1$ vertices, the chordless path on $n \geq 1$ vertices, and the chordless cycle on $n \geq 3$ vertices. As usual, a *tree* is an acyclic connected graph.

A set of pairwise non-adjacent vertices is called *stable*. If S is a stable set, then we denote $N(S) = \{v : N(v) \cap S \neq \emptyset\}$ and $N[S] = N(S) \cup S$. A stable set of maximum size will be referred to as a *maximum stable set* of G . The *stability number* of G , denoted by $\alpha(G)$, is the cardinality of a maximum stable set in G , and $\omega(G) = \alpha(\overline{G})$.

The *disjoint union* of the graphs G_1, G_2 is the graph $G = G_1 \sqcup G_2$ having as vertex set and edge set the disjoint unions of $V(G_1), V(G_2)$ and $E(G_1), E(G_2)$, respectively.

If G_1, G_2 are disjoint graphs, then their *Zykov sum*, (Zykov, [24], [25]), is the graph $G_1 + G_2$ with

$$\begin{aligned} V(G_1 + G_2) &= V(G_1) \cup V(G_2), \\ E(G_1 + G_2) &= E(G_1) \cup E(G_2) \cup \{v_1 v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}. \end{aligned}$$

In particular, $\sqcup nG$ and $+nG$ denote the disjoint union and Zykov sum, respectively, of $n > 1$ copies of the graph G .

A graph G is called *quasi-regularizable* if one can replace each edge of G with a non-negative integer number of parallel copies, so as to obtain a regular multigraph of degree $\neq 0$ (see [3], [4]). Evidently, a disconnected quasi-regularizable graph has no isolated vertices. Moreover, a disconnected graph is quasi-regularizable if and only if any of its connected components spans a quasi-regularizable graph. The following characterization of quasi-regularizable graphs, due to Berge, we shall use in the sequel.

Theorem 1.1 [3] *A graph G is quasi-regularizable if and only if $|S| \leq |N(S)|$ holds for any stable set S of G .*

Let s_k be the number of stable sets in G of cardinality $k \in \{0, 1, \dots, \alpha(G)\}$. The polynomial

$$I(G; x) = s_0 + s_1 x + s_2 x^2 + \dots + s_\alpha x^\alpha, \quad \alpha = \alpha(G),$$

is called the *independence polynomial* of G (Gutman and Harary, [10]). Various properties of this polynomial are presented in a number of papers, like [10], [5], [6], [12], [15], [16], [17], [18], [19], [21].

A finite sequence of real numbers $(a_0, a_1, a_2, \dots, a_n)$ is said to be:

- *unimodal* if there is some $k \in \{0, 1, \dots, n\}$, called the *mode* of the sequence, such that $a_0 \leq \dots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \dots \geq a_n$;
- *log-concave* if $a_i^2 \geq a_{i-1} \cdot a_{i+1}$ holds for $i \in \{1, 2, \dots, n-1\}$.

It is known that any log-concave sequence of positive numbers is also unimodal.

A polynomial is called *unimodal* (*log-concave*) if the sequence of its coefficients is unimodal (log-concave, respectively). For instance, the independence polynomial $I(K_{1,3}; x) = 1 + 4x + 3x^2 + x^3$ is unimodal. However, the independence polynomial of $G = K_{24} + (K_3 \sqcup K_3 \sqcup K_4)$ is not unimodal, since $I(G; x) = 1 + 34x + 33x^2 + 36x^3$ (for other examples, see [1]). Moreover, Alavi *et al.* [1] proved that for any permutation π of $\{1, 2, \dots, \alpha\}$ there is a graph G with $\alpha(G) = \alpha$ such that $s_{\pi(1)} < s_{\pi(2)} < \dots < s_{\pi(\alpha)}$. Nevertheless, for trees, they stated the following still open conjecture.

Conjecture 1.2 [1] *The independence polynomial of a tree is unimodal.*

A graph G is called *well-covered* if all its maximal stable sets have the same cardinality, (Plummer, [22]). If, in addition, G has no isolated vertices and its order equals $2\alpha(G)$, then G is *very well-covered* (Favaron, [8]).

By G^* we mean the graph obtained from G by appending a single pendant edge to each vertex of G , [7]. Let us notice that G^* is well-covered (see, for instance, [13]), and $\alpha(G^*) = n$. In fact, G^* is very well-covered. Moreover, the following result shows that, under certain conditions, any well-covered graph equals G^* for some graph G .

Theorem 1.3 [9] *Let G be a connected graph of girth ≥ 6 , which is isomorphic to neither C_7 nor K_1 . Then G is well-covered if and only if its pendant edges form a perfect matching.*

In other words, Theorem 1.3 shows that, apart from K_1 and C_7 , connected well-covered graphs of girth ≥ 6 are very well-covered. For example, a tree $T \neq K_1$ could be only very well-covered, and this is the case if and only if $T = G^*$ for some tree G (see also [23], [8], [14]).

In [5] it was conjectured that the independence polynomial of any well-covered graph G is unimodal. Michael and Traves [21] proved that this conjecture is true for $\alpha(G) \in \{1, 2, 3\}$, but it is false for $\alpha(G) \in \{4, 5, 6, 7\}$. A family of well-covered graphs with non-unimodal independence polynomials and stability numbers ≥ 8 is presented in [19]. However, the conjecture is still open for very well-covered graphs. In [15] and [16], unimodality of independence polynomials of a number of well-covered trees (e.g., P_n^* , $K_{1,n}^*$) was validated, using the fact that the independence polynomial of a claw-free graph is unimodal (a result due to Hamidoune, [11]). We also showed that $I(G^*; x)$ is unimodal for any G^* whose skeleton G has $\alpha(G) \leq 4$ (see [17]).

Michael and Traves formulated (and verified for well-covered graphs with stability numbers ≤ 7) the following so-called "roller-coaster" conjecture.

Conjecture 1.4 [21] *For any permutation π of the set $\{\lceil \alpha/2 \rceil, \lceil \alpha/2 \rceil + 1, \dots, \alpha\}$, there exists a well-covered graph G , with $\alpha(G) = \alpha$, whose sequence $(s_0, s_1, \dots, s_\alpha)$ satisfies $s_{\pi(\lceil \alpha/2 \rceil)} < s_{\pi(\lceil \alpha/2 \rceil + 1)} < \dots < s_{\pi(\alpha)}$.*

In this paper we prove that if G is a quasi-regularizable graph on $2\alpha(G)$ vertices, then $s_{\lceil (2\alpha(G)-1)/3 \rceil} \geq s_{\lceil (2\alpha(G)-1)/3 \rceil + 1} \geq \dots \geq s_{\alpha(G)}$, while if G is a perfect graph, then $s_{\lceil (\omega(G)-1)/(\omega(G)+1) \rceil} \geq s_{\lceil (\omega(G)-1)/(\omega(G)+1) \rceil + 1} \geq \dots \geq s_\alpha$, where $\alpha = \alpha(G), \omega = \omega(G)$. We infer that for very well-covered graphs, the domain of the roller-coaster conjecture can be shorten to $\{\lceil \alpha/2 \rceil, \lceil \alpha/2 \rceil + 1, \dots, \lceil (2\alpha - 1)/3 \rceil\}$. Moreover, we show that the independence polynomial of a very well-covered graph G is unimodal for $\alpha(G) \leq 9$, and log-concave, whenever $\alpha(G) \leq 5$.

2 Results

In [5] it was shown that $s_{k-1} \leq k \cdot s_k$ and $s_k \leq (n - k + 1) \cdot s_{k-1}, 1 \leq k \leq \alpha(G)$, are true for any well-covered graph G on n vertices.

Proposition 2.1 [21], [18] *If G is a well-covered graph with $\alpha(G) = \alpha$, then the following statements are true:*

- (i) $(\alpha - k) \cdot s_k \leq (k + 1) \cdot s_{k+1}$ holds for $0 \leq k < \alpha$;
- (ii) $s_{k-1} \leq s_k$ for any $1 \leq k \leq (\alpha + 1)/2$.

Notice that Proposition 2.1(i) can fail for non-well-covered graphs, e.g., the graph G_1 in Figure 1 has $\alpha(G_1) = 3$ and $(\alpha(G_1) - 2) \cdot s_2 = 8 > 3 = (2 + 1) \cdot s_3$. However, there are non-well-covered graphs satisfying Proposition 2.1(i), for instance, the graph G_2 in Figure 1. Since $I(G_1; x) = 1 + 6x + 8x^2 + x^3$ and $I(G_2; x) = 1 + 5x + 4x^2$, we see that both G_1 and G_2 satisfy Proposition 2.1(ii). On the other hand, $K_{1,3}$ does not agree with Proposition 2.1(ii), because $\alpha(K_{1,3}) = 3, I(K_{1,3}; x) = 1 + 4x + 3x^2 + x^3$, while $s_1 = 4 > 3 = s_2$.

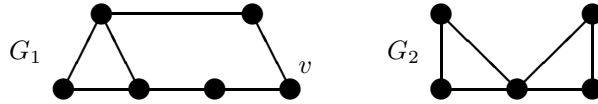


Figure 1: Non-well-covered graph.

Corollary 2.2 *If G is a well-covered graph with $\alpha(G) = \alpha$, then $s_k \leq s_{\alpha-k}$ for $0 \leq k \leq \alpha/2$.*

Proof. Let $k \in \{0, \dots, \lfloor \alpha/2 \rfloor\}$. According to Proposition 2.1(i), we obtain successively, that:

$$\begin{aligned} (\alpha - k) \cdot s_k &\leq (k + 1) \cdot s_{k+1}, \\ (\alpha - k - 1) \cdot s_{k+1} &\leq (k + 2) \cdot s_{k+2}, \\ &\dots \\ (k + 1) \cdot s_{\alpha-k-1} &\leq (\alpha - k) \cdot s_{\alpha-k}. \end{aligned}$$

By multiplying these inequalities, we get

$$\begin{aligned} & (\alpha - k)(\alpha - k - 1)(\alpha - k - 2) \cdots (k + 1) \cdot s_k \cdot s_{k+1} \cdot s_{k+2} \cdots s_{\alpha-k-1} \\ & \leq (k + 1)(k + 2)(k + 3) \cdots (\alpha - k) \cdot s_{k+1} \cdot s_{k+2} \cdot s_{k+3} \cdots s_{\alpha-k-1} \cdot s_{\alpha-k}, \end{aligned}$$

which clearly leads to $s_k \leq s_{\alpha-k}$. ■

The above Corollary 2.2 fails for some non-well-covered graphs, e.g., $K_{1,3}$ has $\alpha(K_{1,3}) = 3$, while $s_1 = 4 > 3 = s_2 = s_{\alpha-1}$. Nevertheless, P_5 is not a well-covered graph, but $I(P_5; x) = 1 + 5x + 6x^2 + x^3$ shows that P_5 satisfies Corollary 2.2.

For a graph G of order n and having $\alpha(G) = \alpha$, we denote

$$\omega_{\alpha-k} = \max\{n - |N[S]| : S \text{ is a stable set with } |S| = k\}, 0 \leq k \leq \alpha.$$

Clearly, $\omega_0 = 0, \omega_\alpha = n$. While $\omega_1(G) \leq \omega(G)$, it is not necessary that $\omega_1(G) = \omega(G)$. For instance, the graph K_3^* (depicted in Figure 2) has $\omega_1 = 2, \omega(K_3^*) = 3$. It is worth mentioning that for any odd chordless cycle $C_{2n+1}, n \geq 2$, or even chordless path $P_{2n}, n \geq 2$, these two parameters are identical.

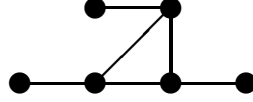


Figure 2: The graph K_3^* .

Lemma 2.3 *If G is a graph of order $n \geq 1$ with $\alpha(G) = \alpha$, then*

$$(k + 1) \cdot s_{k+1} \leq \omega_{\alpha-k} \cdot s_k, 0 \leq k < \alpha,$$

$$\text{in particular, } \alpha \cdot s_\alpha \leq \omega_1 \cdot s_{\alpha-1} \leq \omega(G) \cdot s_{\alpha-1}.$$

Proof. Let $H = (\mathcal{A}, \mathcal{B}, \mathcal{W})$ be the bipartite graph defined as follows: $X \in \mathcal{A} \Leftrightarrow X$ is a stable set in G of size k , then $Y \in \mathcal{B} \Leftrightarrow Y$ is a stable set in G of size $k + 1$, and $XY \in \mathcal{W} \Leftrightarrow X \subset Y$ in G . Since any $Y \in \mathcal{B}$ has exactly $k + 1$ subsets of size k , it follows that $|\mathcal{W}| = (k + 1) \cdot s_{k+1}$. On the other hand, if $X \in \mathcal{A}$ and, then $X \cup \{v\} \in \mathcal{B}$ for any $v \in V(G) - N[X]$, i.e., X has at most $\omega_{\alpha-k}$ neighbors in \mathcal{B} . Hence, we get that $(k + 1) \cdot s_{k+1} = |\mathcal{W}| \leq \omega_{\alpha-k} \cdot |\mathcal{A}| = \omega_{\alpha-k} \cdot s_k$. In particular, for $k = \alpha - 1$, we obtain $\alpha \cdot s_\alpha \leq \omega_1 \cdot s_{\alpha-1} \leq \omega(G) \cdot s_{\alpha-1}$. ■

Let us remark that there are quasi-regularizable graphs with non-unimodal independence polynomials, e.g.,

(a) $G = K_{10} + \sqcup 6K_1$ is connected and has

$$I(G; x) = (1 + x)^6 + 10x = 1 + 16x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6;$$

(b) $G = (K_{24} + \sqcup 6K_1) \sqcup (K_{25} + \sqcup 6K_1)$ is disconnected and has

$$\begin{aligned} I(G; x) &= \left((1 + x)^6 + 24x \right) \left((1 + x)^6 + 25x \right) \\ &= 1 + 61x + 960x^2 + 955x^3 + 1475x^4 + 1527x^5 \\ &\quad + 1218x^6 + 841x^7 + 495x^8 + 220x^9 + 66x^{10} + 12x^{11} + x^{12}. \end{aligned}$$

Proposition 2.4 *If G is a quasi-regularizable graph of order $n = 2\alpha(G) = 2\alpha$, then*

- (i) $\omega_{\alpha-k} \leq 2(\alpha - k), 0 \leq k \leq \alpha$;
- (ii) $(k+1) \cdot s_{k+1} \leq 2(\alpha - k) \cdot s_k, 0 \leq k < \alpha$;
- (iii) $s_{\lceil (2\alpha-1)/3 \rceil} \geq \dots \geq s_{\alpha-1} \geq s_\alpha$.

Proof. (i) Let S be a stable set in G of size $k \geq 0$. According to Theorem 1.1, it follows that $|S| \leq |N(S)|$, which implies $2 \cdot |S| \leq |S \cup N(S)| = |N[S]|$ and, hence, $2 \cdot (\alpha - k) = 2 \cdot (\alpha - |S|) \geq n - |N[S]|$, because $n = 2\alpha$. Consequently, we infer that $\omega_{\alpha-k} \leq 2(\alpha - k)$.

(ii) The result follows by combining Lemma 2.3 and part (i).

(iii) The fact that $(k+1) \cdot s_{k+1} \leq 2(\alpha - k) \cdot s_k$ implies that $s_{k+1} \leq s_k$ holds for $k+1 \geq 2(\alpha - k)$, i.e., for $k \geq (2\alpha - 1)/3$. ■

There are no quasi-regularizable graphs G of order $n > 2\alpha(G)$ that satisfy Proposition 2.4(i),(ii), since for $k = 0$, each of them demands $n \leq 2\alpha(G)$.

In addition, for the graphs G_1, G_2 in Figure 3, $I(G_1; x) = 1 + 6x + 8x^2$ and $I(G_2; x) = 1 + 8x + 19x^2 + 12x^3$ show that Proposition 2.4(iii) is sometimes, but not always, valid for a quasi-regularizable graph G on $n > 2\alpha(G)$ vertices. Notice that G_1 is also well-covered, but not very well-covered.

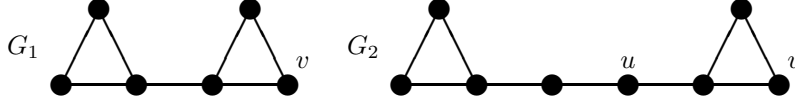


Figure 3: G_1, G_2 are quasi-regularizable graphs, but only G_1 is well-covered.

The graph G in Figure 4 is very well-covered and its independence polynomial $I(G; x) = 1 + 12x + 52x^2 + 110x^3 + 123x^4 + 70x^5 + 16x^6$ is not only unimodal but log-concave, as well.

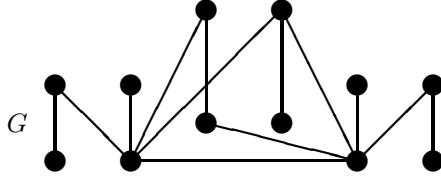


Figure 4: A very well-covered graph with a log-concave independence polynomial.

Theorem 2.5 *If G is a very well-covered graph of order $n \geq 2$ with $\alpha(G) = \alpha$, then*

- (i) $(\alpha - k) \cdot s_k \leq (k+1) \cdot s_{k+1} \leq 2(\alpha - k) \cdot s_k, 0 \leq k < \alpha$;
- (ii) $s_0 \leq s_1 \leq \dots \leq s_{\lceil \alpha/2 \rceil}$ and $s_{\lceil (2\alpha-1)/3 \rceil} \geq \dots \geq s_{\alpha-1} \geq s_\alpha$;
- (iii) $s_{\alpha-2} \cdot s_\alpha \leq s_{\alpha-1}^2$, where $\alpha \geq 2$;
- (iv) $I(G; x)$ is unimodal, while $\alpha \leq 9$;
- (v) $I(G; x)$ is log-concave, while $\alpha \leq 5$.

Proof. (i) It follows from Proposition 2.1(i) and Proposition 2.4, because any well-covered graph without isolated vertices is quasi-regularizable (see Berge, [3], [4]).

(ii) The assertion is established according to Proposition 2.1(ii) and Proposition 2.4.

(iii) Taking $k = \alpha - 2$ in Proposition 2.1(i), we get $2 \cdot s_{\alpha-2} \leq (\alpha - 1) \cdot s_{\alpha-1}$, while substituting $k = \alpha - 1$ in part (i) assures that $\alpha \cdot s_{\alpha} \leq 2 \cdot s_{\alpha-1}$, which together lead to $2\alpha \cdot s_{\alpha-2} \cdot s_{\alpha} \leq 2(\alpha - 1) \cdot s_{\alpha-1}^2$ and, hence, $s_{\alpha-2} \cdot s_{\alpha} \leq s_{\alpha-1}^2$.

(iv) By part (ii), $s_0 \leq s_1 \leq \dots \leq s_{\lceil \alpha/2 \rceil}$ and $s_{\lceil (2\alpha-1)/3 \rceil} \geq \dots \geq s_{\alpha-1} \geq s_{\alpha}$. In addition, the fact that $\alpha(G) \leq 9$ ensures that either $\lceil \alpha/2 \rceil - \lceil (2\alpha - 1)/3 \rceil \leq 1$.

(v) Notice that $s_0 \cdot s_2 = |E(\overline{G})| \leq |V(G)|^2 = s_1^2$ is true for any graph G with $\alpha(G) = \alpha \geq 2$. By part (iii), $s_{\alpha-2} \cdot s_{\alpha} \leq s_{\alpha-1}^2$. Therefore, we have to check that $s_{k-1} \cdot s_{k+1} \leq s_k^2$ only for $k \in \{2, \alpha - 2\}$.

Part (i) implies that $(\alpha - k + 1) \cdot s_{k-1} \leq k \cdot s_k$ and $(k + 1) \cdot s_{k+1} \leq 2(\alpha - k) \cdot s_k$, which together give

$$(\alpha - k + 1) \cdot (k + 1) \cdot s_{k-1} s_{k+1} \leq 2(\alpha - k) \cdot k \cdot s_k^2.$$

If $(\alpha - k + 1) \cdot (k + 1) \geq 2(\alpha - k) \cdot k$, then $s_{k-1} \cdot s_{k+1} \leq s_k^2$. In other words, we are interested to know when $k^2 - \alpha k + \alpha + 1 \geq 0$, while $2 \leq k \leq \alpha - 2$. Since the roots of $k^2 - \alpha k + \alpha + 1$ are $k_{1,2} = (\alpha \pm \sqrt{\alpha^2 - 4\alpha - 4})/2$, we conclude the following, depending on α :

(a) $\alpha \leq 4$, then $\alpha^2 - 4\alpha - 4 < 0$ and, hence, $k^2 - \alpha k + \alpha + 1 \geq 0$ is valid for any k ;

(b) $\alpha = 5$, then $k_1 = 2, k_2 = 3$, and $k^2 - \alpha k + \alpha + 1 \geq 0$ is still true for any k ;

(c) $\alpha \geq 6$, then $k^2 - \alpha k + \alpha + 1 \geq 0$ only for $k = 1$ and $k = \alpha - 1$, because $2 < (\alpha - \sqrt{\alpha^2 - 4\alpha - 4})/2 < 4$ and $2(\alpha - 2) < (\alpha + \sqrt{\alpha^2 - 4\alpha - 4})/2 < 2(\alpha - 1)$.

In summary, the log-concavity condition $s_{k-1} \cdot s_{k+1} \leq s_k^2, 1 \leq k \leq \alpha - 1$, holds for $\alpha \leq 5$. ■

A graph G is called *perfect* if $\chi(H) = \omega(H)$ for any induced subgraph H of G , where $\chi(H)$ denotes the chromatic number of H (Berge, [2]). Lovász proved the theorem claiming that a graph G is perfect if and only if $|V(H)| \leq \alpha(H) \cdot \omega(H)$ for any induced subgraph H of G (see [20]).

Proposition 2.6 *If G is a perfect graph with $\alpha(G) = \alpha$ and $\omega = \omega(G)$, then $s_{\lceil (\omega\alpha-1)/(\omega+1) \rceil} \geq \dots \geq s_{\alpha-1} \geq s_{\alpha}$.*

Proof. Let S be a stable set in G of size $k \geq 0$. Then $H = G - N[S]$ is an induced subgraph of G and has $\alpha(H) \leq \alpha - k$. Therefore, by Lovász's theorem,

$$|V(H)| \leq \omega(H) \cdot \alpha(H) \leq \omega(H) \cdot (\alpha - k) \leq \omega \cdot (\alpha - k)$$

and, hence, $\omega_{\alpha-k} \leq \omega \cdot (\alpha - k)$. Further, according to Lemma 2.3, we obtain that $(k + 1) \cdot s_{k+1} \leq \omega \cdot (\alpha - k) \cdot s_k, 0 \leq k < \alpha$. Now, $s_{k+1} \leq s_k$ is true while $k + 1 \geq \omega \cdot (\alpha - k)$, i.e., for $k \geq (\omega\alpha - 1)/(\omega + 1)$. ■

In fact, in Proposition 2.6 there is some k such that $\lceil (\omega\alpha - 1)/(\omega + 1) \rceil \leq k < \alpha$ if and only if $\alpha - \frac{1+\alpha}{1+\omega} \leq \alpha - 1$, i.e., for $\alpha \geq \omega$. It is worth mentioning that, for general graphs, Lemma 2.3 assures that if a graph G satisfies $\omega(G) \leq \alpha = \alpha(G)$, then $s_\alpha \leq s_{\alpha-1}$. However, the inverse assertion is not true, e.g., $\alpha(K_4 - e) = 2 < 3 = \omega(K_4 - e)$ and $I(K_4 - e; x) = 1 + 4x + x^2$, where by $K_4 - e$ we mean the graph obtained from K_4 by deleting one of its edges.

For non-perfect graphs, Proposition 2.6 is not necessarily false, for example, $I(C_7; x) = 1 + 7x + 14x^2 + 7x^3$. However, the graph $G = \sqcup 4C_5$ is not perfect, $\alpha(G) = 8, \omega(G) = 2$ and

$$I(\sqcup 4C_5; x) = (1 + 5x + 5x^2)^4 = 1 + 20x + 170x^2 + 800x^3 + 2275x^4 + 4000x^5 + \mathbf{4250}x^6 + 2500x^7 + 625x^8$$

is log-concave, but it does not satisfies Proposition 2.6, since $\lceil (\omega\alpha - 1)/(\omega + 1) \rceil = \lceil (2 \cdot 8 - 1)/(2 + 1) \rceil = 5$ and $s_5 = 4000 < 4250 = s_6$.

Any minimal imperfect graph G , i.e., $G = C_{2n+1}, n \geq 2$, or $G = \overline{C_{2n+1}}, n \geq 2$, is claw-free and, consequently, its $I(G; x)$ is log-concave, by Hamidoune's theorem, [11]. However, there are non-perfect graphs, whose independence polynomials are not unimodal, e.g., the disconnected graph $G = (K_{95} + \sqcup 4K_3) \sqcup C_5$ has

$$\begin{aligned} I(G; x) &= (1 + 107x + 54x^2 + 108x^3 + 81x^4) (1 + 5x + 5x^2) \\ &= 1 + 112x + 594x^2 + \mathbf{913}x^3 + 891x^4 + \mathbf{945}x^5 + 405x^6. \end{aligned}$$

Let $H = K_{97} + \sqcup 4K_3$, and G be the graph obtained from H by adding an edge from a vertex of K_{97} to a vertex of some C_5 . Then G is a connected imperfect graph whose $I(G; x)$ is not unimodal, since

$$\begin{aligned} I(G; x) &= (1 + 109x + 54x^2 + 108x^3 + 81x^4) (1 + 4x + 3x^2) \\ &\quad + x(1 + 2x) (1 + 108x + 54x^2 + 108x^3 + 81x^4) \\ &= 1 + 114x + 603x^2 + \mathbf{921}x^3 + 891x^4 + \mathbf{945}x^5 + 405x^6. \end{aligned}$$

Since any bipartite graph G is perfect and has $\omega(G) \leq 2$, we obtain the following result.

Corollary 2.7 *If G is a bipartite graph with $\alpha(G) = \alpha \geq 1$, then $s_{\lceil (2\alpha-1)/3 \rceil} \geq \dots \geq s_{\alpha-1} \geq s_\alpha$.*

In particular, we infer a similar result for trees, whose importance is significant vis-à-vis the conjecture of Alavi *et al.*

Corollary 2.8 *If T is a tree with $\alpha(T) = \alpha$, then $s_{\lceil (2\alpha-1)/3 \rceil} \geq \dots \geq s_{\alpha-1} \geq s_\alpha$.*

For non-bipartite graphs, Corollary 2.7 is not necessarily false (see the graphs in Figure 3).

3 Conclusions

In this paper we prove that for very well-covered graphs the "chaotic interval" $(s_{\lceil \alpha/2 \rceil}, s_{\lceil \alpha/2 \rceil + 1}, \dots, s_\alpha)$ involved in the roller-coaster conjecture of Michael and Traves can be shorten to $(s_{\lceil \alpha/2 \rceil}, s_{\lceil \alpha/2 \rceil + 1}, \dots, s_{\lceil (2\alpha-1)/3 \rceil})$. It seems that one can get even deeper results, by using more efficiently the power of the new defined parameter ω_k .

We also conclude with the two following conjectures sharpening the conjectures of Brown *et al.* and Alavi *et al.* respectively.

Conjecture 3.1 $I(G; x)$ is log-concave for any very well-covered graph G .

Conjecture 3.2 $I(T; x)$ is log-concave for any (well-covered) tree T .

References

- [1] Y. Alavi, P. J. Malde, A. J. Schwenk, P. Erdős, *The vertex independence sequence of a graph is not constrained*, Congressus Numerantium **58** (1987) 15-23.
- [2] C. Berge, *Färbung von Graphen deren sämtliche bzw. deren ungerade Kreise starr sind (Zusammenfassung)*, Wiss.Z. Martin-Luther-Univ. Halle **10** (1961) 114-115.
- [3] C. Berge, *Some common properties for regularizable graphs, edge-critical graphs and B-graphs*, in: Graph Theory and Algorithms Lecture Notes in Computer Science **108** (1980) 108-123, Springer-Verlag, Berlin.
- [4] C. Berge, *Some common properties for regularizable graphs, edge-critical graphs and B-graphs*, Annals of Discrete Mathematics **12** (1982) 31-44.
- [5] J. I. Brown, K. Dilcher, R. J. Nowakowski, *Roots of independence polynomials of well-covered graphs*, Journal of Algebraic Combinatorics **11** (2000) 197-210.
- [6] J. I. Brown, C. A. Hickman, R. J. Nowakowski, *On the location of roots of independence polynomials*, Journal of Algebraic Combinatorics **19** (2004) 273-282.
- [7] R. Dutton, N. Chandrasekharan, R. Brigham, *On the number of independent sets of nodes in a tree*, Fibonacci Quarterly **31** (1993) 98-104.
- [8] O. Favaron, *Very well-covered graphs*, Discrete Mathematics **42** (1982) 177-187.
- [9] A. Finbow, B. Hartnell and R. J. Nowakowski, *A characterization of well-covered graphs of girth 5 or greater*, Journal of Combinatorial Theory B **57** (1993) 44-68.

- [10] I. Gutman, F. Harary, *Generalizations of the matching polynomial*, Utilitas Mathematica **24** (1983) 97-106.
- [11] Y. O. Hamidoune, *On the number of independent k -sets in a claw-free graph*, Journal of Combinatorial Theory B **50** (1990) 241-244.
- [12] C. Hoede, X. Li, *Clique polynomials and independent set polynomials of graphs*, Discrete Mathematics **125** (1994) 219-228.
- [13] V. E. Levit, E. Mandrescu, *Well-covered and König-Egerváry graphs*, Congressus Numerantium **130** (1998) 209-218.
- [14] V. E. Levit, E. Mandrescu, *Well-covered trees*, Congressus Numerantium **139** (1999) 101-112.
- [15] V. E. Levit, E. Mandrescu, *On well-covered trees with unimodal independence polynomials*, Congressus Numerantium Congressus Numerantium **159** (2002) 193-202.
- [16] V. E. Levit, E. Mandrescu, *On unimodality of independence polynomials of some well-covered trees*, DMTCS 2003 (C. S. Calude *et al.* eds.), LNCS **2731**, Springer-Verlag (2003) 237-256.
- [17] V. E. Levit, E. Mandrescu, *A Family of Well-Covered Graphs with Unimodal Independence Polynomials*, Congressus Numerantium **165** (2003) 195-207.
- [18] V. E. Levit, E. Mandrescu, *On the Roots of Independence Polynomials of Almost All Very Well-Covered Graphs*, Los Alamos Archive, pre-print arXiv:math. CO/0305227, 2003, 17 pp.
- [19] V. E. Levit, E. Mandrescu, *Independence polynomials of well-covered graphs: generic counterexamples for the unimodality conjecture*, Los Alamos Archive, pre-print arXiv:math. CO/0309151, 2003, 10 pp.
- [20] L. Lovász, *A characterization of perfect graphs*, Journal of Combinatorial Theory Series B **13** (1972) 95-98.
- [21] T. S. Michael, W. N. Traves, *Independence sequences of well-covered graphs: non-unimodality and the Roller-Coaster conjecture*, Graphs and Combinatorics **19** (2003) 403-411.
- [22] M. D. Plummer, *Some covering concepts in graphs*, Journal of Combinatorial Theory **8** (1970) 91-98.
- [23] G. Ravindra, *Well-covered graphs*, J. Combin. Inform. System Sci. **2** (1977) 20-21.
- [24] A. A. Zykov, *On some properties of linear complexes*, Math. Sb. **24** (1949) 163-188 (in Russian).
- [25] A. A. Zykov, *Fundamentals of graph theory*, BCS Associates, Moscow, 1990.